Generalized Gaussian Quadratures and Singular Value Decompositions of Integral Operators

Vladimir Rokhlin and Norman Yarvin

Research Report YALEU/DCS/RR-1109
May 1996

YALE UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE

Generalized Gaussian quadratures appear to have been introduced by Markov [11,12] late in the last century, and have been studied in great detail as a part of modern analysis (see [2,8,9]). They have not been widely used as a computational tool, in part due to absence of effective numerical schemes for their construction. Recently, a numerical scheme was introduced for the design of such quadratures (see [10]); numerical results presented in [10] indicate that such quadratures dramatically reduce the computational cost of the evaluation of integrals under certain conditions. In this paper, we modify the approach of [10], improving the stability of the scheme and extending its range of applicability. The performance of the method is illustrated with several numerical examples.

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The authors were supported in part by AFOSR under Grant F49620-93-1-0575, in part by ONR under Grants N00014-89-J-1527 and N00014-96-1-0188, and in part by a fellowship from the Fannie and John Hertz Foundation.

Keywords: *Quadratures, Singular Value Decompositions, Chebyshev Systems, Fast Algorithms.*

1 Introduction

Generalized Gaussian quadratures appear to have been introduced by Markov [11, 12] late in the last century. More recent expositions include those by Krein [9] and Karlin [8]. Those expositions contain proofs of the existence of such quadratures for wide classes of functions; however they do not describe a numerical procedure for obtaining the quadrature weights and nodes.

Recently, a paper by Ma et al. [10] described a numerical algorithm for obtaining such quadratures. In [10], a version of Newton's method is introduced for the determination of nodes and weights of generalized Gaussian quadratures. The procedure of [10] guarantees the convergence of the Newton algorithm provided it is started sufficiently close to the solution (whose existence is proven in [11, 9, 8]), and utilizes a continuation procedure to provide such starting points. The present paper describes a variation of that algorithm, which consists mainly of two major changes. The first change is that an entirely different continuation scheme is used; with the new continuation scheme, the algorithm is considerably more robust. The second change is the addition of a preprocessing step which, given as input a large class of functions, uses the singular value decomposition to produce a set of basis functions suitable for the algorithm.

Since a substantial fraction of the algorithm is changed, this paper is written as a repetition of [10], rather than as a list of changes; however, the portions dealing with quadratures for functions with end-point singularities are omitted.

This paper is organized in the following manner. Section 2 summarizes the necessary material from [9] and [8]. Section 3 briefly describes certain standard numerical tools used by the algorithm. Section 4 contains various analytical results to be used in the construction of the algorithm. Section 5 describes the algorithm in detail. Finally, Section 6 contains several numerical examples; the actual nodes and weights obtained in Section 6 are listed in Tables 1-14.

2 Mathematical Preliminaries

We start by introducing some notation. Given a finite sequence of real numbers $x_1 \leq x_2 \leq \cdots \leq x_n$, let the sequence $m_1, \ldots, m_n$ be defined as follows.

$$
\begin{aligned}
m_1 &= 0, & & \\
m_j &= 0 & & \text{if } j > 1 \text{ and } x_j \neq x_{j-1}, \\
m_j &= j-1 & & \text{if } j > 1 \text{ and } x_j = x_{j-1} = \ldots = x_1, \\
m_j &= k & & \text{if } j > k+1 \text{ and } x_j = x_{j-1} = \ldots = x_{j-k} \neq x_{j-k-1}.
\end{aligned}
\tag{1}
$$

2.1 Chebyshev systems

Definition 2.1 *A sequence of functions $\phi_1, \ldots, \phi_k$ will be referred to as a* Chebyshev system *on $[a, b]$ if each of them is continuous and the determinant*

$$
\begin{vmatrix}
\phi_1(x_1) & \cdots & \phi_1(x_n) \\
\vdots & & \vdots \\
\phi_n(x_1) & \cdots & \phi_n(x_n)
\end{vmatrix}
\tag{2}
$$

is nonzero for any sequence of points $x_1, \ldots, x_n$ such that $a \leq x_1 < x_2 \ldots < x_n \leq b$.

An alternate definition of a Chebyshev system is that any linear combination of the functions with nonzero coefficients has no more than n zeros.

A related definition is that of an extended Chebyshev system.

Definition 2.2 *Given a set of functions $\phi_1, \ldots, \phi_n$ which are continuously differentiable on an interval $[a, b]$, and given a sequence of points $x_1, \ldots, x_n$ such that $a \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq b$, let the sequence $m_1, \ldots, m_n$ be defined by (1). Let the matrix $C(x_1, \ldots, x_n) = [c_{ij}]$ be defined by the formula*

$$
c_{ij} = \frac{d^{m_j} \phi_i}{dx^{m_j}}(x_j)
\tag{3}
$$

in which $\frac{d^0 \phi_i}{dx^0}(x_j)$ is taken to be the function value. Then $\phi_1, \ldots, \phi_n$ will be referred to as an extended Chebyshev system *on $[a, b]$ if the determinant $|C(x_1, \ldots, x_n)|$ is nonzero for all such sequences x_i.*

Remark 2.1 *It is obvious from Definition 2.2 that an extended Chebyshev system is a special case of the Chebyshev system. The additional constraint is that the successive points x_i at which*

the function is sampled to form the matrix may be identical; in that case, for each duplicated point, the first corresponding column contains the function values, the second column contains the first derivatives of the functions, the third column contains the second derivatives of the functions, and so forth; this matrix must also be nonsingular.

Examples of Chebyshev and extended Chebyshev systems include the following (additional examples can be found in [8]).

Example 2.1 *The powers* $1, x, x^2, \ldots, x^n$ *form an extended Chebyshev system on the interval* $(-\infty, \infty)$.

Example 2.2 *The exponentials* $e^{-\lambda_1 x}, e^{-\lambda_2 x}, \ldots, e^{-\lambda_n x}$ *form an extended Chebyshev system for any* $\lambda_1, \ldots, \lambda_n > 0$ *on the interval* $[0, \infty)$.

Example 2.3 *The functions* $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx$ *form a Chebyshev system on the interval* $x \in [0, 2\pi)$.

2.2 Generalized Gaussian quadratures

The quadrature rules considered in this paper are expressions of the form

$$\sum_{j=1}^{n} w_j \phi(x_j) \tag{4}$$

where the points $x_j \in \mathbf{R}$ and coefficients $w_j \in \mathbf{R}$ are referred to as the nodes and weights of the quadrature, respectively. They serve as approximations to integrals of the form

$$\int_a^b \phi(x)\omega(x)dx \tag{5}$$

where ω is a non-negative function to be referred to as the weight function.

Quadratures are typically chosen so that the quadrature (4) is equal to the desired integral (5) for some set of functions, commonly polynomials of some fixed order. Of these, the classical Gaussian quadrature rules consist of n nodes and integrate polynomials of order $2n-1$ exactly; these quadratures are used in this paper as a numerical tool (see section 3.2). In [10], the notion of a Gaussian quadrature was generalized as follows:

Definition 2.3 *A quadrature formula will be referred to as* Gaussian *with respect to a set of $2n$ functions $\phi_1, \ldots, \phi_{2n} : [a,b] \to \mathbf{R}$ and a weight function $\omega : [a,b] \to \mathbf{R}^+$, if it consists of n weights and nodes, and integrates the functions ϕ_i exactly with the weight function ω for all $i = 1, \ldots, 2n$. The weights and nodes of a Gaussian quadrature will be referred to as Gaussian weights and nodes respectively.*

The following theorem appears to be due to Markov [11, 12]; proofs of it can also be found in [9] and [8].

Theorem 2.1 *Suppose that the functions $\phi_1, \ldots, \phi_{2n} : [a,b] \to \mathbf{R}$ form a Chebyshev system on $[a,b]$. Suppose in addition that $\omega : [a,b] \to \mathbf{R}$ is non-negative, and is nonzero at more than $n-1$ points on $[a,b]$. Then there exists a unique Gaussian quadrature for $\phi_1, \ldots, \phi_{2n}$ on $[a,b]$ with respect to the weight function ω. The weights of this quadrature are positive.*

2.3 Total positivity

A concept closely related to that of an extended Chebyshev system is that of a extended totally positive (ETP) kernel:

Definition 2.4 *Given a function $K : [a,b] \times [c,d] \to \mathbf{R}$ which is n times continuously differentiable, and given a sequence of points $x_1, \ldots, x_n$ such that $c \le x_1 \le x_2 \le \ldots \le x_n \le d$, let the sequence $m_1, \ldots, m_n$ be defined by (1). Let the functions $\phi_1, \ldots, \phi_n$ be defined by the formula*

$$\phi_j(t) = \frac{\partial^{m_j} K}{\partial x^{m_j}}(x_j, t), \tag{6}$$

in which $\frac{\partial^0 K}{\partial x^0}(x_j, t)$ is taken to be the function value. Then K will be referred to as extended totally positive *if the functions $\phi_1, \ldots, \phi_n$ form an extended Chebyshev system on $[c,d]$ for all such sequences of x_i.*

Examples of ETP kernels include the following (additional examples can be found in [8]).

Example 2.4 *The function e^{-xt} is extended totally positive for $x, t \in [0, \infty)$.*

Example 2.5 *The function $e^{-(x-t)^2}$ is extended totally positive for $x, t \in (-\infty, \infty)$.*

Example 2.6 *The function $1/(x+t)$ is extended totally positive for $x, t \in (0, \infty)$*

A proof of the following lemma can be found in, for example, [8].

Lemma 2.2 *Suppose that K and L are extended totally positive functions of two variables. Then the function M defined by the formula*

$$M(x,t) = \int_c^d K(x,s)L(s,t)ds \tag{7}$$

is extended totally positive. In other words, if the kernels of two integral operators are extended totally positive, the kernel of the product of the two operators is extended totally positive.

The following theorem can be found in [8, 7].

Theorem 2.3 *Suppose that $K : [a,b] \times [a,b] \to \mathbf{R}$ is an extended totally positive kernel. Then the first p eigenfunctions of the integral operator $T : L^2[a,b] \to L^2[a,b]$ defined by the formula*

$$(T\phi)(x) = \int_a^b K(x,s)\phi(s)ds \tag{8}$$

constitute an extended Chebyshev system, for any $p \geq 1$.

3 Numerical Preliminaries

3.1 Newton's Method

In this section we discuss two well-known numerical techniques: Newton's method and the continuation method. A more detailed discussion of these techniques can be found, for example, in [14].

Newton's method is an iterative method for the solution of nonlinear systems of equations of the form $F(x) = 0$, where $F : \mathbf{R}^n \to \mathbf{R}^n$ is a continuously differentiable function of the form

$$F(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}, \tag{9}$$

and $x = (x_1, \ldots, x_n)^T$. The method uses the Jacobian matrix J of F, which is defined by the formula

$$J(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \cdots & \frac{\partial f_n}{\partial x_n}(x) \end{pmatrix} \tag{10}$$

Lemma 3.1 *Suppose that*

$$F(y) = 0 \tag{11}$$

with $F : \mathbf{R}^n \to \mathbf{R}^n$ defined by (9), and that $|J(y)| \neq 0$, with $|J(y)|$ denoting the determinant of the matrix $J(x)$ defined in (10) at the point y. Given a starting point $y_0 \in \mathbf{R}^n$, let the sequence $y_1, y_2, \ldots$ be defined by the formula

$$y_{k+1} = y_k - J^{-1}(y_k)F(y_k). \tag{12}$$

Then there exists a positive real number ε such that for any y_0 satisfying the inequality $||y_0 - y|| < \varepsilon$, the sequence (12) converges to y quadratically, that is, there exists a positive real number α such that

$$||y_{k+1} - y|| < \alpha ||y_k - y||^2. \tag{13}$$

3.1.1 Continuation method

In order for Newton's method to converge, the starting point which is provided to it must be close to the desired solution. One scheme for generating such starting points is the continuation method, which is as follows.

Suppose that in addition to the function $F : \mathbf{R}^n \to \mathbf{R}^n$ whose zero is to be found, another function $G : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ is available which possesses the following properties:

- 1. For any $x \in \mathbf{R}^n$,

$$G(1, x) = F(x). \tag{14}$$

- 2. The solution of the equation

$$G(0, x) = 0 \tag{15}$$

  is known.

- 3. For all $t \in [0,1]$, the equation

$$G(t,x) = 0 \tag{16}$$

has a unique solution x such that the conditions of Lemma 3.1 are satisfied.

- 4. The solution x is a continuous function of t.

If these conditions are met, an algorithm for the solution of $F(x) = 0$ is as follows. Let the points t_i, for $i = 1, \ldots, m$, be defined by the formula $t_i = i/m$. Solve in succession the equations

$$\begin{aligned} G(t_1, x) &= 0, \\ G(t_2, x) &= 0, \\ &\vdots \\ G(t_m, x) &= 0 \end{aligned} \tag{17}$$

using Newton's method, with the starting point for Newton's method for each equation taken to be the solution of the preceding equation. The solution x of the final equation $G(t_m, x) = 0$ is, by (14), identical to the solution of the desired equation $F(x) = 0$. Obviously, for sufficiently large m, Newton's method is guaranteed by Lemma 3.1 to converge at each step.

Remark 3.1 *In practice, it is desirable to choose the smallest m for which the above algorithm will work, in order to reduce the computational cost of the scheme. On the other hand, the largest step $t_i - t_{i-1}$ for which the Newton method will converge commonly varies as a function of t. Thus, in this paper, we use an adaptive version of the scheme.*

3.2 Gaussian integration and interpolation

Classical Gaussian quadrature rules are a well-known numerical tool (see for instance [14]); they integrate polynomials of order $2n - 1$ exactly with respect to some weight function, and consist of n weights and nodes. A variety of Gaussian quadratures were analyzed in the last century, each being defined by a distinct weight function. Of these, the algorithm presented in this paper uses only the Gaussian quadratures for the weight function $\omega(x) = 1$ on the region

of integration $[-1,1]$. These quadratures are closely associated with the Legendre polynomials; we will refer to their nodes as Legendre nodes.

Another numerical tool used in this paper is polynomial interpolation on Legendre nodes. Interpolation refers to the following problem: given two finite real sequences $f_1,\ldots,f_n \in \mathbf{R}$ and $x_1,\ldots,x_n \in [a,b]$, construct a function $f:[a,b] \to \mathbf{R}$ such that $f(x_i) = f_i$ for all $i = 1,\ldots,n$. An interpolation scheme is referred to as linear if the function f depends linearly on the values f_i. One linear interpolation scheme is polynomial interpolation, in which the interpolating function f is a polynomial of degree $n-1$. As is well-known, such a polynomial always exists and is unique. However, in general two numerical difficulties arise with polynomial interpolation using polynomials of high order. The first is that for many sequences of points x_i, the values of the interpolating polynomial between the points x_i are not well-conditioned as a function of the values f_i to be interpolated. The second is that even for those sequences of points where the computation of the values of the interpolating polynomial is well-conditioned, the computation of the coefficients of the power series of the interpolating polynomial is extremely ill-conditioned.

As is well-known, these difficulties do not arise if the points x_i are taken to be Chebyshev nodes and the interpolating polynomial is computed as a series of Chebyshev polynomials rather than as a power series. As the following lemma shows, the difficulties also do not arise if the points x_i are taken to be Legendre nodes and the interpolating polynomial is computed as a series of Legendre polynomials. The lemma makes use of the following properties of the Legendre polynomials: first, that the i'th Legendre polynomial P_i has degree i; second, that the polynomials P_i form an orthonormal system of functions on $[-1,1]$.

Lemma 3.2 *Suppose that $x_1,\ldots,x_n \in [-1,1]$ are the Legendre nodes of order n, and that $w_1,\ldots,w_n \in \mathbf{R}$ are the associated Gaussian weights. Given a sequence $f_1,\ldots,f_n \in \mathbf{R}$, let $p:[-1,1] \to \mathbf{R}$ be the interpolating polynomial of degree $n-1$ such that $p(x_i) = f_i$ for all $i = 1,\ldots,n$, and let $c_0,\ldots,c_{n-1}$ be the coefficients of the Legendre series of p; that is,*

$$p(x) = \sum_{i=0}^{n-1} c_i P_i(x), \tag{18}$$

where $P_i(x)$ is the i'th Legendre polynomial. Then the following relation holds:

$$\sum_{i=1}^{n} w_i f_i^2 = \int_{-1}^{1} p(x)^2 dx = \sum_{i=0}^{n-1} c_i^2. \tag{19}$$

Proof. The second equality of (19) follows from (18) and the orthonormality of the Legendre polynomials. The first equality may be proven as follows: the polynomial p has degree $n-1$, thus its square has degree $2n-2$. Since the Gaussian quadrature integrates exactly all polynomials up to order $2n-1$, it integrates p^2 exactly; thus the first equality of (19) holds. □

3.3 Singular value decomposition

The singular value decomposition (SVD) is a ubiquitous tool in numerical analysis, which is given for the case of real matrices by the following lemma (see, for instance, [3] for more details).

Lemma 3.3 *For any $n \times m$ real matrix A, there exists an $n \times p$ matrix U with orthonormal columns, an $m \times p$ matrix V with orthonormal columns, and a $p \times p$ real diagonal matrix $S = [s_{ij}]$ whose diagonal entries are non-negative, such that $A = USV^*$ and that $s_{ii} < s_{i+1,i+1}$ for all $i = 1, \ldots, p-1$.*

The diagonal entries s_i of S are called singular values; the columns of the matrix V are called right singular vectors; the columns of the matrix U are called left singular vectors.

3.4 Singular value decomposition of integral operators

This section, which follows [5], contains an existence theorem for a factorization of integral operators. The operators $T : L^2[c,d] \to L^2[a,b]$ to which it applies are of the form

$$(Tf)(x) = \int_c^d K(x,t) f(t) dt. \tag{20}$$

in which the function $K : [a,b] \times [c,d] \to \mathbf{R}$ is referred to as the kernel of the operator T. Throughout this section, it will be assumed that all functions are square-integrable; the term "norm" will mean the L^2 norm.

The following theorem, which defines the factorization, is proven in a more general form as Theorem VI.17 in [13].

Theorem 3.4 *Suppose that the function $K : [a,b] \times [c,d] \to \mathbf{R}$ is square integrable. Then there exist two orthonormal sequences of functions $u_i : [a,b] \to \mathbf{R}$ and $v_i : [c,d] \to \mathbf{R}$ and a sequence $s_i \in \mathbf{R}$, for $i = 1, \ldots, \infty$, such that*

$$K(x,t) = \sum_{i=1}^{\infty} u_i(x) s_i v_i(t) \tag{21}$$

and that $s_1 \geq s_2 \geq \ldots \geq 0$. The sequence s_i is uniquely determined by K. Furthermore, the functions v_i are eigenfunctions of the operator T^*T, where T is defined by (20), and the values s_i are the square roots of the eigenvalues of T^*T.

By analogy to the finite-dimensional case, we will refer to this factorization as the singular value decomposition. We will refer to the functions u_i as left singular functions of K (or of T), to v_i as right singular functions, and to s_i as singular values.

As is the case for the discrete singular value decomposition, this decomposition can be used to construct an approximation to the function K, by discarding small singular values and the associated singular functions:

$$K(x,t) \simeq \sum_{i=1}^{p} u_i(x) s_i v_i(t). \tag{22}$$

The error of this approximation can then be computed from (21):

$$K(x,t) - \sum_{i=1}^{p} u_i(x) s_i v_i(t) = \sum_{i=p+1}^{\infty} u_i(x) s_i v_i(t), \tag{23}$$

and, therefore,

$$\left|\left| K(x,t) - \sum_{i=1}^{p} u_i(x) s_i v_i(t) \right|\right| = \sqrt{\sum_{i=p+1}^{\infty} s_i^2}. \tag{24}$$

Using (24), we will be approximating integrals

$$\int_a^b K(x,t)\omega(x)dx \tag{25}$$

by the formula

$$\begin{aligned} \int_a^b K(x,t)\omega(x)dx &\simeq \int_a^b \sum_{i=1}^{p} u_i(x) s_i v_i(t) \omega(x) dx \\ &\simeq \sum_{i=1}^{p} s_i v_i(t) \int_a^b u_i(x)\omega(x)dx. \end{aligned} \tag{26}$$

Thus, a quadrature which is exact for each of the integrals

$$\int_a^b u_i(x)\omega(x)dx, \tag{27}$$

for $i = 1, \ldots, p$, is an approximate quadrature for integrals of the form (25).

The following theorem shows that if an operator is extended totally positive, its singular functions form an extended Chebyshev system.

Theorem 3.5 *Suppose that $K : [a,b] \times [c,d] \to \mathbf{R}$ is extended totally positive. Then the first p left singular functions of K form an extended Chebyshev system, for any p; likewise the first p right singular functions of K form an extended Chebyshev system, for any p.*

Proof. Let the integral operator $T : L^2[c,d] \to L^2[a,b]$ be defined by the formula

$$(Tf)(x) = \int_c^d K(x,t)f(t)dt, \tag{28}$$

and the function $L : [a,b] \to [a,b]$ be defined by the formula

$$L(x,t) = \int_c^d K(x,s)K(t,s)ds. \tag{29}$$

Clearly, the integral operator $S : L^2[a,b] \to L^2[a,b]$ defined by the formula $S = T^*T$ has the kernel L:

$$\begin{aligned}(S\phi)(x) &= \int_a^b \int_c^d K(x,s)K(t,s)ds\phi(t)dt \\ &= \int_a^b L(x,t)\phi(t)dt. \end{aligned}\tag{30}$$

Since K is extended totally positive, due to Lemma 2.2, L is also extended totally positive. Thus, by Theorem 2.3, the eigenfunctions of S constitute an extended Chebyshev system. By Theorem 3.4, these eigenfunctions are identical to the left singular functions of T, which proves that the first p left singular functions of T constitute an extended Chebyshev system, for any p. The proof for the right singular functions is identical. □

4 Analytical Apparatus

4.1 Convergence of Newton's method

In this section, we observe that the nodes and the weights of a Gaussian quadrature satisfy a certain system of nonlinear equations. We then prove that the Newton method for this system of equations is always quadratically convergent, provided the functions to be integrated constitute an extended Chebyshev system.

Given a set of functions $\phi_1, \ldots, \phi_{2n}$ and a weight function ω, the Gaussian quadrature is defined by the system of equations

$$\begin{aligned} \sum_{j=1}^{n} w_i \phi_1(x_i) &= \int_a^b \phi_1(x)\omega(x)dx, \\ \sum_{j=1}^{n} w_i \phi_2(x_i) &= \int_a^b \phi_2(x)\omega(x)dx, \\ &\vdots \\ \sum_{j=1}^{n} w_i \phi_{2n}(x_i) &= \int_a^b \phi_{2n}(x)\omega(x)dx \end{aligned} \tag{31}$$

(see Definition 2.3). Let the left hand sides of these equations be denoted by f_1 through f_{2n}. Then each f_i is a function of the weights $w_1, \ldots, w_n$ and nodes $x_1, \ldots, x_n$ of the quadrature. Its partial derivatives are given by the obvious formulae

$$\frac{\partial f_k}{\partial w_i} = \phi_k(x_i), \tag{32}$$

$$\frac{\partial f_k}{\partial x_i} = w_i \phi_k'(x_i). \tag{33}$$

Thus, the Jacobian matrix of the system (31) is

$$J(x_1, \ldots, x_n, w_1, \ldots, w_n) = \begin{pmatrix} \phi_1(x_1) & \cdots & \phi_1(x_n) & w_1\phi_1'(x_1) & \cdots & w_n\phi_1'(x_n) \\ \vdots & & \vdots & \vdots & & \vdots \\ \phi_{2n}(x_1) & \cdots & \phi_{2n}(x_n) & w_1\phi_{2n}'(x_1) & \cdots & w_n\phi_{2n}'(x_n) \end{pmatrix}. \tag{34}$$

Lemma 4.1 *Suppose that the functions $\phi_1, \ldots, \phi_{2n}$ form an extended Chebyshev system. Let the Gaussian quadrature for these functions be denoted by $\hat{w}_i$ and $\hat{x}_i$. Then the determinant of J is nonzero at the point which constitutes the Gaussian quadrature; in other words, $|J(\hat{x}_1, \ldots, \hat{x}_n, \hat{w}_1, \ldots, \hat{w}_n)| \neq 0$.*

Proof. It is immediately obvious from (34) that

$$J(\hat{x}_1, \ldots, \hat{x}_n, \hat{w}_1, \ldots, \hat{w}_n)| =$$
$$\hat{w}_1 \cdot \hat{w}_2 \cdot \cdots \cdot \hat{w}_{n-1} \cdot \hat{w}_n \begin{vmatrix} \phi_1(x_1) & \cdots & \phi_1(x_n) & \phi_1'(x_1) & \cdots & \phi_1'(x_n) \\ \vdots & & \vdots & \vdots & & \vdots \\ \phi_{2n}(x_1) & \cdots & \phi_{2n}(x_n) & \phi_{2n}'(x_1) & \cdots & \phi_{2n}'(x_n) \end{vmatrix}. \tag{35}$$

If $\phi_1, \ldots, \phi_{2n}$ form an extended Chebyshev system, then by Theorem 2.1, the weights $\hat{w}_1, \ldots, \hat{w}_n$ of the Gaussian quadrature are positive. In addition, by the definition of an extended Chebyshev system, the determinant in the right hand side of (35) is nonzero. Thus

$$|J(\hat{x}_1, \ldots, \hat{x}_n, \hat{w}_i, \ldots, \hat{w}_i)| \neq 0. \tag{36}$$

□

Corollary 4.2 *Under the conditions of Lemma 4.1, the Gaussian weights and nodes depend continuously on the weight function.*

4.2 Linear interpolation

Given a collection of n points $x_1, x_2, \ldots, x_n \in [a, b]$, an interpolation scheme with the nodes $x_1, x_2, \ldots, x_n$ is a linear mapping $L : \mathbf{R}^n \to L^\infty[a, b]$ such that for any point $y \in \mathbf{R}^n$,

$$L(y)(x_i) = y_i, \tag{37}$$

for all $i = 1, 2, \ldots, n$. For a function $f \in L^\infty[a, b]$, the error $\delta_L(f)$ of interpolation of the interpolation scheme L is defined by the formula

$$\delta_L(f) = \max_{x \in [a,b]} |f(x) - L(F)(x)| \tag{38}$$

where $F = (f(x_1), f(x_2), \ldots, f(x_n))^T$.

The following lemma serves as a justification for the notation we use in Section 4.3 for linear interpolation schemes.

Lemma 4.3 *Suppose $L : \mathbf{R}^n \to L^2[a,b]$ is a linear interpolation scheme with n nodes $x_1, \ldots, x_n \in [a,b]$. Then there exists a sequence of functions $\alpha_1, \ldots, \alpha_n : [a,b] \to \mathbf{R}$ such that for any vector $f \in \mathbf{R}^n$, with elements $f = (f_1, \ldots, f_n)^T$,*

$$(Lf)(x) = \sum_{i=1}^{n} f_i \alpha_i(x), \tag{39}$$

for all $x \in [a,b]$.

Proof. Let the vectors $e_1, \ldots, e_n \in \mathbf{R}^n$ with elements $e_i = (e_{i1}, \ldots, e_{in})^T$ be the standard basis in $\mathbf{R}^n$; that is, $e_{ii} = 1$ for all $i = 1, \ldots, n$, and $e_{ij} = 0$ for all $i, j = 1, \ldots, n$ such that $i \neq j$. Let the functions $\alpha_1, \ldots, \alpha_n : [a,b] \to \mathbf{R}$ be defined by the formula $\alpha_i = Le_i$. Since the interpolation scheme L is linear, for any vector $f \in \mathbf{R}^n$ with elements $f = (f_1, \ldots, f_n)^T$, and for any point $x \in [a,b]$,

$$\begin{aligned} (Lf)(x) &= \left(L\left(\sum_{i=1}^{n} f_i e_i\right)\right)(x) \\ &= \sum_{i=1}^{n} f_i (Le_i)(x) \\ &= \sum_{i=1}^{n} f_i \alpha_i(x). \end{aligned} \tag{40}$$

□

In the case of polynomial interpolation, the functions α_i are referred to as Lagrange polynomials; by analogy to that case, we will in general refer to the functions α_i as the Lagrange functions of the interpolation scheme.

The following lemma provides an error bound for approximation of a function of two variables using two one-dimensional interpolation formulae, expressed in terms of error bounds for each one-dimensional interpolation scheme applied separately. Its proof is an exercise in elementary analysis, and is omitted.

Lemma 4.4 *Suppose that $x_1, x_2, \ldots, x_n \in [a,b]$ and $t_1, t_2, \ldots, t_m \in [c,d]$ are two finite real sequences, and that $\alpha_1, \alpha_2, \ldots, \alpha_n : [a,b] \to \mathbf{R}$ and $\beta_1, \beta_2, \ldots, \beta_m : [c,d] \to \mathbf{R}$ are two sequences of bounded functions. Suppose further that $L_1 : \mathbf{R}^n \to L^\infty[a,b]$ is an interpolation formula with the nodes $x_1, \ldots, x_n$ and Lagrange functions $\alpha_1, \ldots, \alpha_n$, and $L_2 : \mathbf{R}^m \to L^\infty[c,d]$ is an*

interpolation formula with the nodes $t_1, \ldots, t_m$ *and Lagrange functions* $\beta_1, \ldots, \beta_m$. *Suppose that* $\eta \in \mathbf{R}$ *is such that*

$$\sum_{i=1}^{n} |\alpha_i(x)| < \eta, \tag{41}$$

for all $x \in [a, b]$, *and*

$$\sum_{j=1}^{m} |\beta_j(t)| < \eta, \tag{42}$$

for all $t \in [c, d]$. *Finally, suppose that* K *is a function* $[a, b] \times [c, d] \to \mathbf{R}$, *and that for all* $x \in [a, b]$ *and* $t \in [c, d]$,

$$\left| K(x,t) - \sum_{i=1}^{n} K(x_i, t)\alpha_i(x) \right| < \varepsilon \tag{43}$$

and

$$\left| K(x,t) - \sum_{j=1}^{m} K(x, t_j)\beta_j(t) \right| < \varepsilon. \tag{44}$$

Then

$$\left| K(x,t) - \sum_{i=1}^{n} \sum_{j=1}^{m} K(x_i, t_j)\alpha_i(x)\beta_j(t) \right| < \varepsilon(1 + \eta), \tag{45}$$

for all $x \in [a, b]$ *and* $t \in [c, d]$.

4.3 Approximation of SVD of an integral operator

This section describes a numerical procedure for computing an approximation to the singular value decomposition of an integral operator.

The algorithm uses quadratures which possess the following property.

Definition 4.1 *We will say that the combination of a quadrature and a linear interpolation scheme* preserves inner products *on an interval* $[a, b]$ *if it possesses the following properties.*

- *1. The nodes of the quadrature are identical to the nodes of the interpolation scheme.*
- *2. The quadrature integrates exactly any product of two interpolated functions; that is, for any two functions* $f, g : [a, b] \to \mathbf{R}$ *produced by the interpolation scheme, the integral*

$$\int_a^b f(x)g(x)dx \tag{46}$$

is computed exactly by the quadrature.

Quadratures and interpolation schemes which possess this property include:

Example 4.1 *The combination of a (classical) Gaussian quadrature at Legendre nodes and polynomial interpolation at the same nodes preserves inner products, since polynomial interpolation on* n *nodes produces an interpolating polynomial of order* $n-1$*, the product of two such polynomials is a polynomial of order* $2n-2$*, and a Gaussian quadrature integrates exactly all polynomials up to order* $2n-1$.

Example 4.2 *If an interval is broken into several subintervals, and a quadrature and interpolation scheme which preserves inner products is used on each subinterval, then the arrangement as a whole preserves inner products on the original interval. (This follows directly from the definition.)*

Example 4.3 *The combination of the trapezoidal rule on the interval* $[0, 2\pi]$*, and Fourier interpolation (using the interpolation functions* $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx$*) preserves inner products.*

The algorithm takes as input a function $K : [a, b] \times [c, d] \to \mathbf{R}$. It uses the following numerical tools:

- 1. A quadrature and a linear interpolation scheme on the interval $[a, b]$ which preserve inner products. Let the weights and nodes of this quadrature be denoted by $w_1^x, \ldots, w_n^x \in \mathbf{R}$ and $x_1, \ldots, x_n \in [a, b]$ respectively. Let the Lagrange functions (see Section 4.2) of the interpolation scheme be denoted by $\alpha_1, \ldots, \alpha_n : [a, b] \to \mathbf{R}$.

- 2. A quadrature and a linear interpolation scheme on the interval $[c, d]$ which preserve inner products. Let the weights and nodes of this quadrature be denoted by $w_1^t, \ldots, w_m^t \in \mathbf{R}$ and $t_1, \ldots, t_m \in [c, d]$ respectively. Let the Lagrange functions of the interpolation scheme be denoted by $\beta_1, \ldots, \beta_m : [c, d] \to \mathbf{R}$.

As will be shown below, the accuracy of the algorithm is then determined by the accuracy to which the above two interpolation schemes approximate K.

The output of the algorithm is a sequence of functions $u_1, \ldots, u_p : [a,b] \to \mathbf{R}$, a sequence of functions $v_1, \ldots, v_p : [c,d] \to \mathbf{R}$, and a sequence of singular values $s_1, \ldots, s_p \in \mathbf{R}$, which form an approximation to the singular value decomposition of K.

Description of the algorithm:

- 1. Construct the $n \times m$ matrix $A = [a_{ij}]$ defined by the formula

$$a_{ij} = K(x_i, t_j)\sqrt{w_i^x \cdot w_j^t}. \tag{47}$$

- 2. Compute the singular value decomposition of A, to produce the factorization

$$A = USV^*, \tag{48}$$

  where $U = [u_{ij}]$ is an $n \times p$ matrix with orthonormal columns, $V = [v_{ij}]$ is an $m \times p$ matrix with orthonormal columns, and S is a $p \times p$ diagonal matrix whose j'th diagonal entry is s_j.

- 3. Construct the $n \times p$ matrix $\hat{U} = [\hat{u}_{ij}]$ and the $m \times p$ matrix $\hat{V} = [\hat{v}_{ij}]$ defined by the formulae

$$\hat{u}_{ik} = u_{ik}/\sqrt{w_i^x}, \tag{49}$$

$$\hat{v}_{jk} = v_{jk}/\sqrt{w_j^t}. \tag{50}$$

- 4. For any points $x \in [a,b]$ and $t \in [c,d]$, evaluate the functions $u_k : [a,b] \to \mathbf{R}$ and $v_k : [c,d] \to \mathbf{R}$ via the formulae

$$u_k(x) = \sum_{i=1}^{n} \hat{u}_{ik} \cdot \alpha_i(x), \tag{51}$$

$$v_k(t) = \sum_{j=1}^{m} \hat{v}_{jk} \cdot \beta_j(t), \tag{52}$$

  for all $k = 1, \ldots, p$.

Theorem 4.5 *Suppose that the combination of the quadrature with weights and nodes $w_1^x, \ldots, w_n^x \in \mathbf{R}$ and $x_1, \ldots, x_n \in [a,b]$, respectively, and the interpolation scheme with Lagrange functions $\alpha_1, \ldots, \alpha_n : [a,b] \to \mathbf{R}$, preserves inner products on $[a,b]$.*

Suppose in addition that the combination of the quadrature with weights and nodes $w_1^t, \ldots, w_m^t \in \mathbf{R}$ and $t_1, \ldots, t_m \in [c,d]$, respectively, and the interpolation scheme with Lagrange functions $\beta_1, \ldots, \beta_m : [c,d] \to \mathbf{R}$, preserves inner products on $[c,d]$.

For any function $K : [a,b] \times [c,d] \to \mathbf{R}$, let $u_i : [a,b] \to \mathbf{R}$, $v_i : [c,d] \to \mathbf{R}$, and $s_i \in \mathbf{R}$ be defined in (47)-(52), for all $i = 1, \ldots, p$. Then

- *1. The functions u_i are orthonormal, i.e.*

$$\int_a^b u_i(x) u_k(x) dx = \delta_{ik} \tag{53}$$

for all $i, k = 1, \ldots, p$, with δ_{ik} the Kronecker symbol ($\delta_{ij} = 1$ if $i = j$, 0 otherwise).

- *2. The functions v_i are orthonormal, i.e.*

$$\int_c^d v_i(t) v_k(t) dx = \delta_{ik} \tag{54}$$

for all $i, k = 1, \ldots, p$.

- *3. The function $\tilde{K} : [a,b] \times [c,d] \to \mathbf{R}$ defined by the formula*

$$\tilde{K}(x,t) = \sum_{j=1}^{p} s_j u_j(x) v_j(t), \tag{55}$$

is identical to the function produced by sampling K on the grid of points (x_i, t_j), then interpolating with the two interpolation schemes. That is,

$$\tilde{K}(x,t) = \sum_{i=1}^{n} \sum_{j=1}^{m} K(x_i, t_j) \alpha_i(x) \beta_j(t). \tag{56}$$

Proof. We first prove (56). Combining (51), (52), and (55), we have

$$\begin{aligned}
\tilde{K}(x,t) &= \sum_{k=1}^{p} s_k \left(\sum_{i=1}^{n} u_k(x_i) \alpha_i(x) \right) \left(\sum_{j=1}^{m} v_k(w_j^x) \beta_j(t) \right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\sum_{k=1}^{p} u_k(x_i) s_k v_k(w_j^x) \right) \alpha_i(x) \beta_j(t) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\sum_{k=1}^{p} (u_{ik}/\sqrt{w_i^x}) s_k (v_{jk}/\sqrt{w_j^t}) \right) \alpha_i(x) \beta_j(t)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sum_{k=1}^{p} u_{ik}s_k v_{jk}/\sqrt{w_i^x w_j^t}\right)\alpha_i(x)\beta_j(t) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\left(a_{ij}/\sqrt{w_i^x w_j^t}\right)\alpha_i(x)\beta_j(t). \qquad (57)
\end{aligned}
$$

Now, (56) follows from the combination of (57) and (47).

We now demonstrate the orthonormality of the functions u_i. Since these are functions produced by interpolation, and since the quadrature on $[a,b]$ is assumed to integrate exactly all products of pairs of interpolated functions,

$$
\begin{aligned}
\int_a^b u_i(x)u_k(x)dx &= \sum_{j=1}^{n} w_j^x u_i(x_j)u_k(x_j) \\
&= \sum_{j=1}^{n} w_j^x (u_{ji}/\sqrt{w_j^x})(u_{jk}/\sqrt{w_j^x}) \\
&= \sum_{j=1}^{n} u_{ji}u_{jk}. \qquad (58)
\end{aligned}
$$

Since the last sum in (58) is the inner product of two columns of the orthonormal matrix U (see (48),

$$
\int_a^b u_i(x)u_k(x)dx = \delta_{ik}. \qquad (59)
$$

The orthonormality of the functions v_i is proven in the same manner. □

Remark 4.1 *Obviously, the above proof approximates the singular value decomposition of the operator* $T : L^2[c,d] \to L^2[a,b]$ *with the kernel* K *by constructing an approximation* $\tilde{T}$ *with kernel* $\tilde{K}$ *to the operator* T *that is of finite rank, and constructing the exact singular value decomposition of the latter.*

Observation 4.2 *In the preceding proof, the assumption that each combination of quadrature and interpolation scheme preserves inner products was used only to demonstrate the orthonormality of the corresponding singular functions. Thus, if the conditions of Theorem 4.5 hold, with the exception that the quadrature on* $[a,b]$ *does not preserve inner products, then (54) and (56) hold (but, in general, (53) does not).*

Remark 4.3 *Theorem 4.5 and Lemma 4.4 generalize trivially to higher dimensions. One-dimensional quadratures and interpolation formulae have to be replaced with their multidimensional counterparts; otherwise, the proofs are unchanged.*

5 Numerical Algorithm

This section describes a numerical algorithm for the evaluation of nodes and weights of generalized Gaussian quadratures. The algorithm's input is a sequence of functions $\phi_1, \ldots, \phi_{2n} : [a,b] \to \mathbf{R}$ which form an extended Chebyshev system on $[a,b]$, and a weight function $\omega_1 : [a,b] \to \mathbf{R}^+$. Its output is the weights and nodes of the quadrature. The main components of the algorithm are as follows (not listed in order of execution):

- 1. Newton's method is used to solve (31) which defines the Gaussian quadrature.
- 2. An adaptive version of the continuation method (Section 3.1.1) is used to provide starting points for Newton's method. The continuation scheme used here is different from that used in [10]; the details of the continuation scheme and of the method of adaption are described below.
- 3. The algorithm of section 4.3 can be used as an optional preprocessing step, which takes as input a kernel of an integral operator and produces its singular functions. The first $2n$ of the left singular functions are then used as input to the main algorithm.

5.1 Continuation Scheme

The continuation scheme used is as follows. Let the weight functions $\omega : [0,1] \times [a,b] \to \mathbf{R}^+$ be defined by the formula

$$\omega(\alpha, x) = \alpha\omega_1(x) + (1-\alpha)\sum_{j=1}^{n} \delta(x - c_j), \tag{60}$$

where ω_1 is the weight function for which a Gaussian quadrature is desired, δ denotes the Dirac delta function, and the points $c_j \in [a,b]$ are arbitrary distinct points. These weight functions have the following properties:

- 1. With $\alpha = 1$, the weight function is equal to the desired weight function ω_1, due to (60).

- 2. With $\alpha = 0$, the Gaussian weights and nodes are

$$w_j = 1, \tag{61}$$

$$x_j = c_j, \tag{62}$$

  for $j = 1, \ldots, n$, whatever the functions ϕ_i are (since $\omega(0, x) = 0$, unless $x = c_j$ for some $j \in [1, n]$).

- 3. The quadrature weights and nodes depend continuously on α (by Corollary 4.2).

The intermediate problems which the continuation method solves are the Gaussian quadratures relative to the weight functions $\omega(\alpha, *)$. The scheme starts by setting $\alpha = 0$, then increases α in an adaptive manner until $\alpha = 1$, as follows. A current step size is maintained, by which α is incremented after each successful termination of Newton's method. After each unsuccessful termination of Newton's method, the step size is halved and the algorithm restarts from the point yielded by the last successful termination. After a certain number of successful steps, the current step size is doubled. (Experimentally, the current problem was found to be well suited to an aggressive mode of adaption: in the authors' implementation, the initial value of the step size was chosen to be one, and the step size was doubled after a single successful termination of Newton's method.)

5.1.1 Starting points

The choice of the points c_j was left indefinite above. In exact arithmetic the algorithm would converge for any choice of distinct points (see Lemma 4.1). However the number of steps of the continuation method, and thus the speed of execution, is affected by the choice. More importantly, the numerical stability of the scheme might be compromised due to poor conditioning of the matrix J (see (34)). Indeed, while Lemma 4.1 guarantees that the matrix J is non-singular, it says nothing about its condition number. Thus, in the authors' implementation, the points

c_j used for the production of the quadrature of order n were computed from the nodes x_j of the quadrature of order $n-1$, by the formulae

$$c_1 = x_1, \tag{63}$$

$$c_i = (x_{i-1} + x_i)/2, \qquad i = 2, \ldots, n-1, \tag{64}$$

$$c_n = x_{n-1}. \tag{65}$$

With this choice, no failures to converge have been encountered in the authors' experience.

6 Numerical examples

A variety of quadratures were generated to illustrate the performance of the above algorithm. In each case the preprocessing step of producing singular functions was used. This step requires two sets of quadratures and interpolation schemes, which must approximate the desired kernel to the desired accuracy. These quadratures and interpolation schemes were chosen so that the approximation was accurate to about the precision of the arithmetic that was used. The following combination of quadrature and interpolation scheme which preserves inner products was used: the interval of integration was divided into several subintervals, and a combination of a (classical) Gaussian quadrature at Legendre nodes and polynomial interpolation was used on each subinterval.

In each of the following examples, the calculations were done in extended precision (Fortran `REAL*16`) arithmetic, with the exception of the last example, which was done in double precision (`REAL*8`) arithmetic.

6.1 Exponentials

In this example we construct quadratures for the integral

$$\int_0^\infty e^{-xt} dx, \tag{66}$$

under the condition that $1 \le t \le 500$. In this case, the corresponding kernel $K : [0, \infty) \times [1, 500] \to \mathbf{R}$ is given by

$$K(x,t) = e^{-xt}, \tag{67}$$

and is extended totally positive; thus its singular functions form an extended Chebyshev system. A sample of the quadratures produced by the algorithm is included in Tables 1-3; for double precision accuracy, a 27-point quadrature is required.

6.2 Complex Exponentials

Here, we design quadratures for a new version [5] of the two-dimensional Fast Multipole Method. These quadratures are for the integral

$$\int_0^\infty e^{-xz} dx, \tag{68}$$

under the condition that $z \in \mathbb{C}$ is constrained to lie in the region D of the complex plane which consists of the rectangle $[1,4] \times [-4,4]$ with a 1×1 square deleted from each of its two left hand corners, as depicted in Figure 1. Since both the true integral (equal to $1/z$) and the quadrature which approximates the integral are complex analytic on that region, due to the maximum modulus principle the maximum error of the quadrature is achieved on the boundary δD of the region. Accordingly, the kernel whose singular functions were computed was $K(x,z) = e^{-xz}$, with z varying over δD. A brief examination of the resulting singular functions shows that they do not form a Chebyshev system; if they did so, the i'th function would have $i-1$ zeros, yet it has many more. Thus the algorithm is not guaranteed to work; however, it did so. A sample of the resulting quadratures is included in Tables 4-6; in this case a quadrature yielding double precision accuracy contains 32 nodes.

6.3 Exponentials multiplied by I_0

In this example, quadrature formulae are constructed for integrals of the form

$$\int_0^\infty I_0(xy) e^{-xt} dx, \tag{69}$$

under the condition that $t \in [1, 500]$ and $y \in [0, t-1]$; these formulae were designed to be used in a version of the one-dimensional Fast Multipole Method which is used in an algorithm [6] for the fast Hankel transform. In this case the singular functions produced by the precomputation stage were extremely similar to those for exponentials alone; unlike in the case of complex

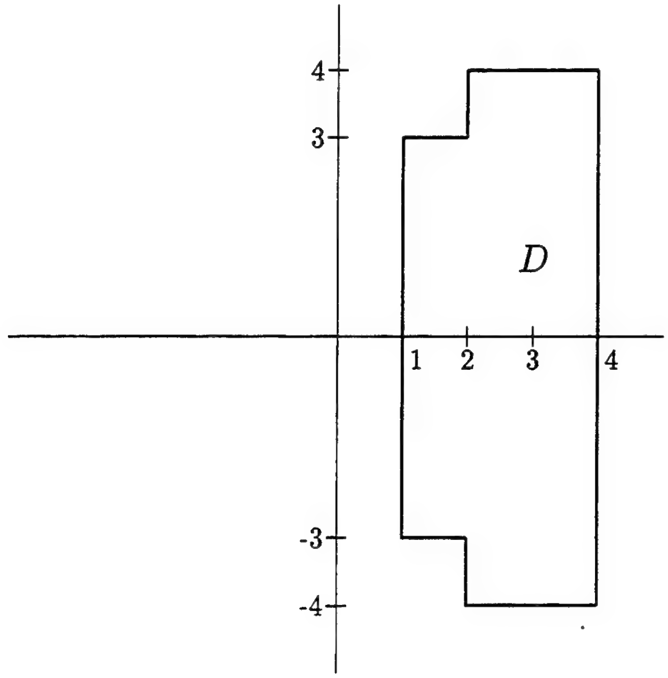

Figure 1: Coefficient z of complex exponentials to be integrated

exponentials, it is possible that they form a Chebyshev system. In any case, the algorithm converged, producing a quadrature which required two more nodes for double precision accuracy than were required for the integration of exponentials alone (i.e. 29 nodes). A sample of the resulting quadratures is included in Tables 7-9.

6.4 Exponentials multiplied by J_0

Here, we construct quadratures for the integral

$$\int_0^\infty J_0(xy)e^{-xt}dx, \tag{70}$$

under the conditions that $t \in [1,4]$ and $y \in [0, 4\sqrt{2}]$, where J_0 denotes the Bessel function of the first kind of order zero. These quadratures are used in a new version [4] of the three-dimensional Fast Multipole Method. J_0 is given by the well-known (see for instance [1]) formula

$$J_0(y) = \frac{1}{\pi}\int_0^\pi \cos(y\cos\theta)d\theta. \tag{71}$$

Substituting (71) into (70) yields the integral

$$\int_0^\infty \left(\frac{1}{\pi} \int_0^\pi \cos(xy\cos\theta) d\theta \right) e^{-xt} dx$$
$$= \frac{1}{\pi} \int_0^\pi \int_0^\infty \cos(xy\cos\theta) e^{-xt} dx d\theta. \tag{72}$$

Thus a quadrature accurate for the integral

$$\int_0^\infty \cos(xy) e^{-xt} dx, \tag{73}$$

under the conditions that $t \in [1,4]$ and $y \in [0, 4\sqrt{2}]$, is also accurate for the integral (70) under the same conditions on y and t. Since the function $\cos(xy)e^{-xt}$ is a harmonic function of y and t, by the maximum modulus principle the maximum error of a quadrature for (73) lies on the boundary δD of the rectangular region $t \in [1,4]$, $y \in [0, 4\sqrt{2}]$. Accordingly, the kernel whose singular functions were computed was $K(x,z) = \cos(xy)e^{-xt}$, with (t,y) varying over δD. As in the case of complex exponentials, the singular functions have too many zeros to form a Chebyshev system, however the algorithm converged. A sample of the resulting quadratures is included in Tables 10-14; for single precision accuracy 22 nodes are required.

6.5 Numerical Observations

The following observations were made in the course of our numerical experiments.

- 1. The number of continuation steps required is highly variable; in many cases, only one step sufficed to produce the quadrature; less frequently, up to fifty or so continuation steps were required. This variability occurred even between quadratures for successive numbers n of nodes, with the same weight function and kernel K.

- 2. The algorithm worked in the cases where Theorem 2.1 applied, and also in cases where it did not. In the latter cases, it is conceivable that the resulting quadratures would have negative weights, or that they would not be unique. However, all computed weights were positive, and, while no systematic attempt was made to look for non-uniqueness of the quadratures, no instance of it was observed.

7 Generalizations and Applications

- 1. The success of the algorithm in instances where Theorem 2.1 does not apply suggests that further theoretical investigation of conditions for the existence of generalized Gaussian quadratures would be profitable.

- 2. An obvious generalization of these results is to quadratures for integrals in more than one dimension. However, such an extension does not seem to have been explored classically; the authors are investigating a generalization of Theorem 2.1 for multidimensional quadratures.

- 3. An obvious application of the algorithm of this paper is for the efficient evaluation of functions represented by their integral transforms (see Sections 6.1, 6.2, 6.3, 6.4 above, as well as [5] and [4]). The method of steepest descent in the numerical complex analysis provides a wide field of applications for such algorithms.

- 4. An entirely different field of applications involves the numerical solution of integral equations with singular kernels; of particular interest are boundary integral equations of scattering theory on regions with corners. The authors are currently pursuing this direction of research.

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Table 1: Quadratures for exponentials

Quadratures for the integral

$$\int_0^\infty e^{-xt}dx,$$

under the condition that $1 \le t \le 500$.

N	Nodes (x_i)	Weights (w_i)	Error
6	0.2934661296034111E-02	0.8078894059616301E-02	0.827E-03
	0.2122706574797170E-01	0.3337852721645502E-01	
	0.8809516681098265E-01	0.1157432569817795E+00	
	0.3048205241689060E+00	0.3589923073929015E+00	
	0.9407821348001514E+00	0.1018252445219498E+01	
	0.2710823671107057E+01	0.2863049428178813E+01	
8	0.2027451178542047E-02	0.5378159157423945E-02	0.726E-04
	0.1244627909236754E-01	0.1689195281391659E-01	
	0.4102057941602644E-01	0.4443355769152601E-01	
	0.1144937922230447E+00	0.1123335780703449E+00	
	0.2954175879426304E+00	0.2711448390160103E+00	
	0.7210624530246545E+00	0.6264263367286485E+00	
	0.1687074064747948E+01	0.1408860943521751E+01	
	0.3896282168946610E+01	0.3344043060866228E+01	
14	0.1075073588251350E-02	0.2783455121689438E-02	0.366E-07
	0.5889243490962496E-02	0.7006395914900820E-02	
	0.1560078432135377E-01	0.1279502133157069E-01	
	0.3258052212086110E-01	0.2192733340131016E-01	
	0.6154351752779967E-01	0.3737740049082059E-01	
	0.1109619891032348E+00	0.6379243969367225E-01	
	0.1951651530857407E+00	0.1084594588227473E+00	
	0.3377699882687942E+00	0.1830223278438481E+00	
	0.5772805419211481E+00	0.3061647832783700E+00	
	0.9761165652290038E+00	0.5079755103629931E+00	
	0.1635615445691163E+01	0.8381174751258640E+00	
	0.2723809484786727E+01	0.1385562498413431E+01	
	0.4541163041303490E+01	0.2347348786059432E+01	
	0.7767616655342678E+01	0.4444622409829190E+01	

Table 2: Quadratures for exponentials (continued)

Quadratures for the integral

$$\int_0^\infty e^{-xt} dx,$$

under the condition that $1 \leq t \leq 500$.

N	Nodes (x_i)	Weights (w_i)	Error
23	0.6351980115825126E-03	0.1635014749191032E-02	0.356E-12
	0.3390398468501349E-02	0.3906005173541682E-02	
	0.8533988111011606E-02	0.6439810761063304E-02	
	0.1642520313864894E-01	0.9442176581726002E-02	
	0.2767253774178896E-01	0.1321157735118120E-01	
	0.4324590958238922E-01	0.1817939708425140E-01	
	0.6462964536146855E-01	0.2494901611240477E-01	
	0.9401198766143719E-01	0.3433118358535485E-01	
	0.1345150809637970E+00	0.4739385869478771E-01	
	0.1904922248317389E+00	0.6554991255546435E-01	
	0.2679364541057184E+00	0.9069257914926255E-01	
	0.3750493014038291E+00	0.1253810727774845E+00	
	0.5230239495194809E+00	0.1730865464562759E+00	
	0.7271139907371750E+00	0.2385227411097447E+00	
	0.1008092183746637E+01	0.3281047583403605E+00	
	0.1394262856335610E+01	0.4506156991375293E+00	
	0.1924307475579603E+01	0.6182455916131797E+00	
	0.2651497602918851E+01	0.8483813375121229E+00	
	0.3650430825998876E+01	0.1167112116649273E+01	
	0.5029133182526411E+01	0.1617207388482339E+01	
	0.6954672288346456E+01	0.2279901680035951E+01	
	0.9721470480335499E+01	0.3352383978313540E+01	
	0.1402158019660932E+02	0.5608355831510393E+01	

Table 3: Quadratures for exponentials (continued)

Quadratures for the integral

$$\int_0^\infty e^{-xt}dx,$$

under the condition that $1 \le t \le 500$.

N	Nodes (x_i)	Weights (w_i)	Error
27	0.5378759010624780E-03	0.1383311204046008E-02	0.323E-14
	0.2860176825815242E-02	0.3279869733166365E-02	
	0.7148658617716300E-02	0.5330932895600203E-02	
	0.1360965515937845E-01	0.7646093110803760E-02	
	0.2257800188133212E-01	0.1037458793227033E-01	
	0.3456421989535069E-01	0.1372178039022047E-01	
	0.5032042618508775E-01	0.1796868836009351E-01	
	0.7092509447124836E-01	0.2348971809947674E-01	
	0.9788439120828463E-01	0.3076860552710760E-01	
	0.1332509921950535E+00	0.4041894092839717E-01	
	0.1797695570864978E+00	0.5321827718681367E-01	
	0.2410654714132133E+00	0.7016094768858448E-01	
	0.3218961915636380E+00	0.9253048536912244E-01	
	0.4284852078938826E+00	0.1219928996130354E+00	
	0.5689615509235298E+00	0.1607156476580828E+00	
	0.7539347736933301E+00	0.2115215602167892E+00	
	0.9972472224438443E+00	0.2780925850550500E+00	
	0.1316964566299846E+01	0.3652478333806065E+00	
	0.1736698582009859E+01	0.4793398853949993E+00	
	0.2287418444638146E+01	0.6288554258416082E+00	
	0.3010034073439038E+01	0.8254021100491956E+00	
	0.3959315495048493E+01	0.1085495633209734E+01	
	0.5210381702393131E+01	0.1434174907278760E+01	
	0.6870768194824406E+01	0.1913323186889750E+01	
	0.9106577764323245E+01	0.2604342790201154E+01	
	0.1221294512896673E+02	0.3708436699287805E+01	
	0.1689348652665484E+02	0.6023086156615004E+01	

Table 4: Quadratures for complex exponentials

Quadratures for the integral

$$\int_0^\infty e^{-xz}\,dx,$$

under the condition that $z \in \mathbb{C}$ lies in the region D of the complex plane depicted in Figure 1.

N	Nodes (x_i)	Weights (w_i)	Error
7	0.1099271618238942E+00	0.2775596224308371E+00	0.107E-02
	0.5491694162336780E+00	0.5900612562744907E+00	
	0.1271416827286341E+01	0.8478610527159362E+00	
	0.2239523056474301E+01	0.1088510946164213E+01	
	0.3446836330005198E+01	0.1323732065869006E+01	
	0.4877666068772302E+01	0.1534838877513932E+01	
	0.6502607915187052E+01	0.1719048349027934E+01	
10	0.7940097370047949E-01	0.2021326824744206E+00	0.398E-04
	0.4059967502704461E+00	0.4452920131070853E+00	
	0.9586054827056690E+00	0.6549257007902238E+00	
	0.1707633862341116E+01	0.8399190894283777E+00	
	0.2634252243120157E+01	0.1012522786957398E+01	
	0.3733067811454947E+01	0.1185698158021533E+01	
	0.5005663556309191E+01	0.1358749093234873E+01	
	0.6447614701968830E+01	0.1523775992304074E+01	
	0.8049956086568744E+01	0.1681530325372958E+01	
	0.9806270415536372E+01	0.1839363349513445E+01	
17	0.4810701202067075E-01	0.1231104634892695E+00	0.156E-07
	0.2505848757761927E+00	0.2802783031579153E+00	
	0.6047137247359728E+00	0.4258197681747696E+00	
	0.1097956904569217E+01	0.5586462093341786E+00	
	0.1718338718562377E+01	0.6804909782597703E+00	
	0.2456116758149593E+01	0.7938674951201829E+00	
	0.3304098340771468E+01	0.9013184216898933E+00	
	0.4257638182548677E+01	0.1005434048854524E+01	
	0.5314792420007071E+01	0.1109016499341237E+01	
	0.6476281670685671E+01	0.1214353975155841E+01	
	0.7744192070244406E+01	0.1321691619491553E+01	
	0.9119761690912472E+01	0.1429331461933850E+01	
	0.1060243019989989E+02	0.1535781285935049E+01	
	0.1219107734541850E+02	0.1641633199457229E+01	
	0.1388654365182865E+02	0.1750408322216728E+01	
	0.1569478151303984E+02	0.1869608472411435E+01	
	0.1762993064234310E+02	0.2013038988665553E+01	

Table 5: Quadratures for complex exponentials (continued)

Quadratures for the integral

$$\int_0^\infty e^{-xz}\,dx,$$

under the condition that $z \in \mathbb{C}$ lies in the region D of the complex plane depicted in Figure 1.

N	Nodes (x_i)	Weights (w_i)	Error
26	0.3186852812707167E-01	0.8168641985324435E-01	0.801E-12
	0.1670696511352561E+00	0.1882692109574778E+00	
	0.4070892206145096E+00	0.2909692905166523E+00	
	0.7472439175044331E+00	0.3884080535348131E+00	
	0.1182020706329825E+01	0.4802140096002388E+00	
	0.1705877918170689E+01	0.5666489183411501E+00	
	0.2313727921958910E+01	0.6483159527750231E+00	
	0.3001165447553827E+01	0.7259502257876793E+00	
	0.3764536084546370E+01	0.8003069049756682E+00	
	0.4600933023898150E+01	0.8721245141158127E+00	
	0.5508181675990506E+01	0.9421323128655350E+00	
	0.6484841722444765E+01	0.1011073349889977E+01	
	0.7530228834188133E+01	0.1079715704282156E+01	
	0.8644426324529564E+01	0.1148813015862887E+01	
	0.9828211695813383E+01	0.1218968645024915E+01	
	0.1108280524626372E+02	0.1290432847974804E+01	
	0.1240944939152817E+02	0.1363002295760370E+01	
	0.1380901099033861E+02	0.1436177127644119E+01	
	0.1528185219684248E+02	0.1509511398129948E+01	
	0.1682806304875190E+02	0.1582954591205827E+01	
	0.1844797932809189E+02	0.1657078657112459E+01	
	0.2014289281830454E+02	0.1733260266399871E+01	
	0.2191599603042421E+02	0.1814015981921987E+01	
	0.2377386163950739E+02	0.1903865269011978E+01	
	0.2572922307071097E+02	0.2011585306477846E+01	
	0.2780603356526977E+02	0.2156295623247961E+01	

Table 6: Quadratures for complex exponentials (continued)

Quadratures for the integral

$$\int_0^\infty e^{-xz}\,dx,$$

under the condition that $z \in \mathbb{C}$ lies in the region D of the complex plane depicted in Figure 1.

N	Nodes (x_i)	Weights (w_i)	Error
32	0.2599836293936463E-01	0.6666723984893712E-01	0.282E-14
	0.1365267137471029E+00	0.1541635906448656E+00	
	0.3335929178284293E+00	0.2395065808061905E+00	
	0.6144848200204676E+00	0.3216997252047818E+00	
	0.9757788966059570E+00	0.4002647717267098E+00	
	0.1413764904033582E+01	0.4750901483689786E+00	
	0.1924760915837073E+01	0.5463258676163205E+00	
	0.2505322024543478E+01	0.6142802064945914E+00	
	0.3152355883303843E+01	0.6793389599054632E+00	
	0.3863171917607857E+01	0.7419135289744640E+00	
	0.4635491817809779E+01	0.8024143819948454E+00	
	0.5467443320214809E+01	0.8612429338113290E+00	
	0.6357552049608270E+01	0.9187946472618356E+00	
	0.7304739365689955E+01	0.9754669332016366E+00	
	0.8308328039815608E+01	0.1031665911277885E+01	
	0.9368051427922781E+01	0.1087805382732233E+01	
	0.1048405410784282E+02	0.1144289073127857E+01	
	0.1165686219043638E+02	0.1201466511491073E+01	
	0.1288729607812993E+02	0.1259564363290567E+01	
	0.1417631366171486E+02	0.1318623693719137E+01	
	0.1552481519724356E+02	0.1378494220804057E+01	
	0.1693348466077443E+02	0.1438913247537656E+01	
	0.1840274326790348E+02	0.1499642611759877E+01	
	0.1993284537567664E+02	0.1560607527964651E+01	
	0.2152409758022736E+02	0.1622001781246697E+01	
	0.2317716791120881E+02	0.1684367617089771E+01	
	0.2489347969157359E+02	0.1748695093705612E+01	
	0.2667574373640977E+02	0.1816615869561691E+01	
	0.2852878296929011E+02	0.1890833030920412E+01	
	0.3046102218822619E+02	0.1976151351204436E+01	
	0.3248756039307094E+02	0.2082162437141457E+01	
	0.3463653566705793E+02	0.2231151873780132E+01	

Table 7: Quadratures for exponentials multiplied by I_0

Quadratures for the integral

$$\int_0^\infty I_0(xy)e^{-xt}dx,$$

under the condition that $t \in [1, 500]$ and $y \in [0, t-1]$.

N	Nodes (x_i)	Weights (w_i)	Error
6	0.2014951814758335E-02	0.5609959210156781E-02	0.997E-03
	0.1524995123116156E-01	0.2486640535654579E-01	
	0.6752356088881507E-01	0.9333052982723889E-01	
	0.2501565900307728E+00	0.3114318476382451E+00	
	0.8236005667135590E+00	0.9419185523143980E+00	
	0.2511917343393000E+01	0.2783935122763889E+01	
8	0.1310754453518395E-02	0.3522290544269296E-02	0.892E-04
	0.8427815046421337E-02	0.1188533050797089E-01	
	0.2934337237542595E-01	0.3342585308732757E-01	
	0.8663331921085451E-01	0.8985909963205312E-01	
	0.2362895983874226E+00	0.2297963615059060E+00	
	0.6083828926140039E+00	0.5600968338362269E+00	
	0.1496074584403707E+01	0.1320244303895297E+01	
	0.3613104405570935E+01	0.3253613362413727E+01	
14	0.6424288534795956E-03	0.1667964367860395E-02	0.900E-07
	0.3562319666990144E-02	0.4298903067080389E-02	
	0.9643424057074440E-02	0.8159545461265918E-02	
	0.2074298599770349E-01	0.1463864640961027E-01	
	0.4057928260022333E-01	0.2614391453322226E-01	
	0.7600572280169251E-01	0.4665755537868725E-01	
	0.1390443053485344E+00	0.8276361628521883E-01	
	0.2503136051566992E+00	0.1454478222995341E+00	
	0.4447622918282108E+00	0.2529871458046016E+00	
	0.7811276346003586E+00	0.4357009925372973E+00	
	0.1357818162257100E+01	0.7446059596729966E+00	
	0.2341992534236977E+01	0.1271434906786924E+01	
	0.4036529413075654E+01	0.2216807353831690E+01	
	0.7126502974662635E+01	0.4302367103374836E+01	

Table 8: Quadratures for exponentials multiplied by I_0 (continued)

Quadratures for the integral

$$\int_0^{\infty} I_0(xy)e^{-xt}dx,$$

under the condition that $t \in [1, 500]$ and $y \in [0, t-1]$.

N	Nodes (x_i)	Weights (w_i)	Error
24	0.3495789092315762E-03	0.9002840624465873E-03	0.925E-12
	0.1870021782900649E-02	0.2160252603038041E-02	
	0.4726267824851739E-02	0.3590338301868660E-02	
	0.9151716313193496E-02	0.5325399369607390E-02	
	0.1554236980028004E-01	0.7559106650189445E-02	
	0.2452696840462340E-01	0.1056758825931061E-01	
	0.3706303150690139E-01	0.1473724813105991E-01	
	0.5456277459785074E-01	0.2059685692839059E-01	
	0.7905495952668643E-01	0.2885941258668471E-01	
	0.1134006713086074E+00	0.4049079881256631E-01	
	0.1615933468425095E+00	0.5680844261155735E-01	
	0.2291729794308585E+00	0.7961469039423471E-01	
	0.3238013639554152E+00	0.1113872397913262E+00	
	0.4560558428504051E+00	0.1555264308649468E+00	
	0.6405113416979851E+00	0.2166942382013035E+00	
	0.8972279324556263E+00	0.3012922008808448E+00	
	0.1253809582199313E+01	0.4181453830530776E+00	
	0.1748318313157252E+01	0.5795680006227947E+00	
	0.2433578887375728E+01	0.8031688787093632E+00	
	0.3384010828873551E+01	0.1115348337169697E+01	
	0.4707795467897619E+01	0.1559213913124744E+01	
	0.6572440958883276E+01	0.2216187240183199E+01	
	0.9272736577094724E+01	0.3282974439738362E+01	
	0.1349943616527142E+02	0.5529398603135539E+01	

Table 9: Quadratures for exponentials multiplied by I_0 (continued)

Quadratures for the integral

$$\int_0^\infty I_0(xy)e^{-xt}dx,$$

under the condition that $t \in [1, 500]$ and $y \in [0, t-1]$.

N	Nodes (x_i)	Weights (w_i)	Error
29	0.2855179413353365E-03	0.7344503079351386E-03	0.299E-14
	0.1519624696728258E-02	0.1744538390662211E-02	
	0.3804359141657344E-02	0.2844687196642974E-02	
	0.7260138000706486E-02	0.4098961298933580E-02	
	0.1208205371062810E-01	0.5593550200298448E-02	
	0.1856564543199398E-01	0.7444670271885530E-02	
	0.2714156753309568E-01	0.9807968524698940E-02	
	0.3842017800878239E-01	0.1288914176031762E-01	
	0.5324783256625659E-01	0.1695687345717790E-01	
	0.7277755829761968E-01	0.2235879759838917E-01	
	0.9855788611173273E-01	0.2954235698585380E-01	
	0.1326465035778468E+00	0.3908423859367898E-01	
	0.1777590387840778E+00	0.5173159700695577E-01	
	0.2374657658898870E+00	0.6845695550893067E-01	
	0.3164509240422835E+00	0.9052903520482935E-01	
	0.4208524457939620E+00	0.1196036182345700E+00	
	0.5587051648321601E+00	0.1578409524693449E+00	
	0.7405185479404663E+00	0.2080593451794129E+00	
	0.9800319873390735E+00	0.2739397750144418E+00	
	0.1295209795621391E+01	0.3603059290242059E+00	
	0.1709570851677607E+01	0.4735231867763476E+00	
	0.2254009385987865E+01	0.6221016600956893E+00	
	0.2969389638669206E+01	0.8176841100086656E+00	
	0.3910476327629495E+01	0.1076831175000068E+01	
	0.5152430007642100E+01	0.1424628439002124E+01	
	0.6802867813529709E+01	0.1902988149814232E+01	
	0.9027979519502084E+01	0.2593285548365225E+01	
	0.1212289908066820E+02	0.3696550722303479E+01	
	0.1679085599535762E+02	0.6009492062220468E+01	

Table 10: Quadratures for exponentials multiplied by J_0

Quadratures for the integral

$$\int_0^\infty J_0(xy)e^{-xt}dx,$$

under the condition that $t \in [1,4]$ and $y \in [0, 4\sqrt{2}]$.

N	Nodes (x_i)	Weights (w_i)	Error
8	0.1093474676900044E+00	0.2710750266277354E+00	0.162E-02
	0.5176974101534121E+00	0.5276915884394641E+00	
	0.1133065916111916E+01	0.6915150441387948E+00	
	0.1881350151107404E+01	0.7983440040645204E+00	
	0.2717854096012053E+01	0.8716416012135397E+00	
	0.3616502749074490E+01	0.9264383911692414E+00	
	0.4562710533038212E+01	0.9729462225948307E+00	
	0.5549008853485283E+01	0.1024138658446855E+01	
12	0.7685522448236467E-01	0.1937803229242497E+00	0.709E-04
	0.3802271685596512E+00	0.4024780894501363E+00	
	0.8629501667245919E+00	0.5551232854865536E+00	
	0.1477406574242533E+01	0.6684012296815303E+00	
	0.2190593072512602E+01	0.7541446224405415E+00	
	0.2979188555054684E+01	0.8203905361353097E+00	
	0.3826805213168235E+01	0.8731017778158731E+00	
	0.4722181214285143E+01	0.9167109597437153E+00	
	0.5657828852278510E+01	0.9545728875259875E+00	
	0.6629008403962641E+01	0.9893709749159459E+00	
	0.7632911519449263E+01	0.1023874368056413E+01	
	0.8669258567695921E+01	0.1067824933823433E+01	

Table 11: Quadratures for exponentials multiplied by J_0 (continued)

Quadratures for the integral

$$\int_0^\infty J_0(xy)e^{-xt}dx,$$

under the condition that $t \in [1,4]$ and $y \in [0, 4\sqrt{2}]$.

N	Nodes (x_i)	Weights (w_i)	Error
21	0.4557110658309451E-01	0.1162693279863745E+00	0.553E-07
	0.2345692419777160E+00	0.2587357630437822E+00	
	0.5560507435597863E+00	0.3805968474435821E+00	
	0.9888622621190326E+00	0.4818526125575090E+00	
	0.1514051750681985E+01	0.5659864564983776E+00	
	0.2116308405894669E+01	0.6365120448492290E+00	
	0.2783466404955423E+01	0.6961876640755223E+00	
	0.3505774922334249E+01	0.7471226714069135E+00	
	0.4275358227777114E+01	0.7909873646366727E+00	
	0.5085850421343891E+01	0.8291470860681295E+00	
	0.5932129958898720E+01	0.8627341849390908E+00	
	0.6810110232637652E+01	0.8926928974293094E+00	
	0.7716569940856932E+01	0.9198161427100997E+00	
	0.8649018954485772E+01	0.9447809312411800E+00	
	0.9605599035641322E+01	0.9681847744461964E+00	
	0.1058501999437337E+02	0.9905857720543664E+00	
	0.1158653269800637E+02	0.1012551797552972E+01	
	0.1260993904708917E+02	0.1034729712476950E+01	
	0.1365563090220990E+02	0.1057973242328095E+01	
	0.1472471197434301E+02	0.1083960111883219E+01	
	0.1582111587898742E+02	0.1123223099240344E+01	

Table 12: Quadratures for exponentials multiplied by J_0 (continued)

Quadratures for the integral

$$\int_0^\infty J_0(xy)e^{-xt}dx,$$

under the condition that $t \in [1, 4]$ and $y \in [0, 4\sqrt{2}]$.

N	Nodes (x_i)	Weights (w_i)	Error
31	0.3135427831034307E-01	0.8024339887513055E-01	0.195E-10
	0.1633283571233953E+00	0.1826480678501762E+00	
	0.3938519939248281E+00	0.2767843166469057E+00	
	0.7134977521472219E+00	0.3607878783606646E+00	
	0.1112086865038666E+01	0.4347972857765461E+00	
	0.1580107134125432E+01	0.4998566920961263E+00	
	0.2109237828913374E+01	0.5572245708288816E+00	
	0.2692380266283717E+01	0.6080577723352450E+00	
	0.3323499813712884E+01	0.6533243161908575E+00	
	0.3997438592537748E+01	0.6938153813251570E+00	
	0.4709757745640057E+01	0.7301846396770468E+00	
	0.5456619026749223E+01	0.7629849054856576E+00	
	0.6234697842382201E+01	0.7926941021324604E+00	
	0.7041117206272996E+01	0.8197316250932437E+00	
	0.7873394668593335E+01	0.8444685010430712E+00	
	0.8729397582414611E+01	0.8672343080184358E+00	
	0.9607304495127067E+01	0.8883227316567794E+00	
	0.1050557184595927E+02	0.9079967283507420E+00	
	0.1142290582410390E+02	0.9264937318864332E+00	
	0.1235823954272516E+02	0.9440311133071873E+00	
	0.1331071586594676E+02	0.9608120820932964E+00	
	0.1427967645292451E+02	0.9770323352089769E+00	
	0.1526465797627137E+02	0.9928880149352948E+00	
	0.1626539716056547E+02	0.1008585996949826E+01	
	0.1728184746281412E+02	0.1024358393992377E+01	
	0.1831421215592517E+02	0.1040484810479409E+01	
	0.1936300165981823E+02	0.1057329638354610E+01	
	0.2042912676709522E+02	0.1075408955912917E+01	
	0.2151404706129260E+02	0.1095542344669040E+01	
	0.2262012600158427E+02	0.1119613828562103E+01	
	0.2375360482790972E+02	0.1159632125663126E+01	

Table 13: Quadratures for exponentials multiplied by J_0 (continued)

Quadratures for the integral

$$\int_0^\infty J_0(xy)e^{-xt}dx,$$

under the condition that $t \in [1,4]$ and $y \in [0, 4\sqrt{2}]$.

N	Nodes (x_i)	Weights (w_i)	Error
40	0.2450466782681923E-01	0.6278240289055225E-01	0.147E-13
	0.1282174857176023E+00	0.1441452770046284E+00	
	0.3113726824539808E+00	0.2213018947962683E+00	
	0.5689058522320320E+00	0.2927436383298983E+00	
	0.8947716487384897E+00	0.3579417172934986E+00	
	0.1282740805986842E+01	0.4170055991300023E+00	
	0.1726889168240672E+01	0.4703924681663025E+00	
	0.2221829818237509E+01	0.5186914865873826E+00	
	0.2762775128318641E+01	0.5624975493703959E+00	
	0.3345508771449190E+01	0.6023527423690355E+00	
	0.3966321542990838E+01	0.6387284183350381E+00	
	0.4621940741778159E+01	0.6720275534119337E+00	
	0.5309466588865855E+01	0.7025949580565075E+00	
	0.6026319872772099E+01	0.7307287944249567E+00	
	0.6770200467500875E+01	0.7566905396059194E+00	
	0.7539054647612656E+01	0.7807125897154367E+00	
	0.8331048881923509E+01	0.8030036866777486E+00	
	0.9144548183658569E+01	0.8237527017133555E+00	
	0.9978097658074295E+01	0.8431313322224179E+00	
	0.1083040639472408E+02	0.8612961564381806E+00	
	0.1170033322583332E+02	0.8783903510592598E+00	
	0.1258687411609641E+02	0.8945452599331251E+00	
	0.1348915109370842E+02	0.9098819208500799E+00	
	0.1440640271349045E+02	0.9245126105589752E+00	
	0.1533797609080401E+02	0.9385424474160881E+00	
	(Continued in the next table)		

Table 14: Quadratures for exponentials multiplied by J_0 (continued)

Quadratures for the integral

$$\int_0^\infty J_0(xy)e^{-xt}dx,$$

under the condition that $t \in [1,4]$ and $y \in [0, 4\sqrt{2}]$.

N	Nodes (x_i)	Weights (w_i)	Error
40	(Continued from the preceding table)		0.147E-13
	0.1628332058271154E+02	0.9520710907702972E+00	
	0.1724198323818701E+02	0.9651945915710470E+00	
	0.1821360621166772E+02	0.9780074909877168E+00	
	0.1919792646634440E+02	0.9906053454228810E+00	
	0.2019477834484074E+02	0.1003088034610363E+01	
	0.2120410011858627E+02	0.1015564634343247E+01	
	0.2222594665494397E+02	0.1028161001682954E+01	
	0.2326051184537432E+02	0.1041032846271463E+01	
	0.2430816659609159E+02	0.1054383431852286E+01	
	0.2536951778065644E+02	0.1068492599609564E+01	
	0.2644549219792192E+02	0.1083742894323642E+01	
	0.2753746108815434E+02	0.1100713259554107E+01	
	0.2864745668803461E+02	0.1120408914213389E+01	
	0.2977833401798621E+02	0.1144615248532595E+01	
	0.3093837103779525E+02	0.1182108938213342E+01	